Karlsruhe Institute of Technology

Institute for Applied Materials

Plasticity

Lecture for “Mechanical Engineering” and “Materials Science and Engineering”

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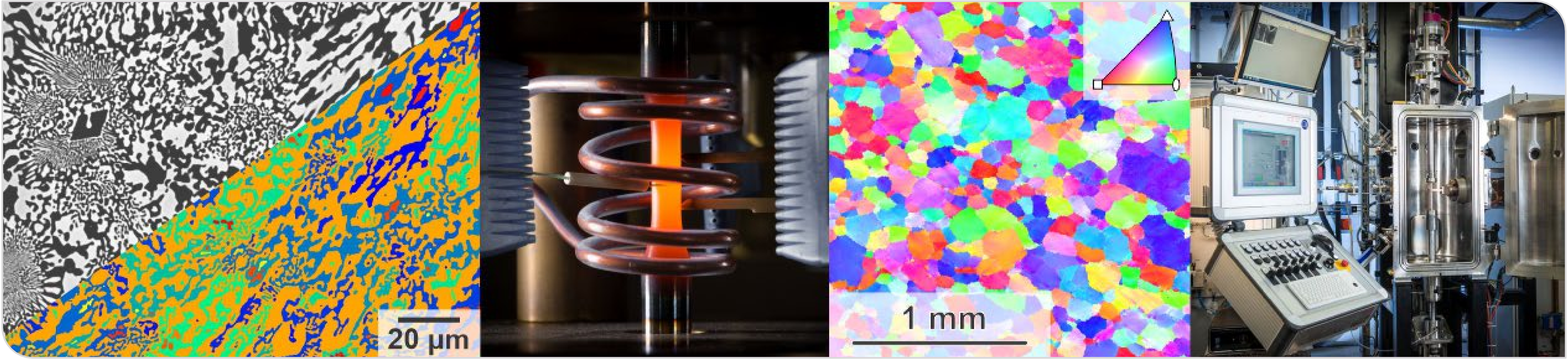


Topics

- Revision of Elasticity Theory
  - Ultra Short Revision of Tensor Algebra
  - Displacement Vector
  - Strain Tensor, Exact Derivation and Small Strain Approximation
  - Stress Tensor, Derivation Incl. Continuum Interpretation
  - Linear Elasticity
  - Anisotropy

Topics

- The quantification stresses and properties of dislocations request simple mathematical methods of the description of deformed solids. **The simplest approximation is linear elasticity of continuous solids**.
- The **mathematical framework is based on tensor algebra** which is briefly revised on the following slides.
- We will not only utilize rotations of coordinate systems in this chapter; we will use the approaches sometimes during the lecture.

Revision of Tensor Algebra, Definitions

- Scalar product of vectors (rank one):

$$\boldsymbol{a} \cdot \boldsymbol{b} = a\, b\, \cos \sphericalangle(\boldsymbol{a}, \boldsymbol{b})$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i\, \boldsymbol{e_i} \cdot b_k\, \boldsymbol{e_k} \overset{\text{assoc.}}{\triangleq} a_i\, b_k\, \boldsymbol{e_i} \cdot \boldsymbol{e_k} \overset{\text{orthon.}}{\triangleq} a_i\, b_k\, \delta_{ik} = a_i\, b_i$$

- Cross product of vectors:

$$\boldsymbol{a} \times \boldsymbol{b} = a\, b\, \sin \sphericalangle(\boldsymbol{a}, \boldsymbol{b})\, \boldsymbol{n}$$

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = a_i\, \boldsymbol{e_i} \times b_k\, \boldsymbol{e_k} \overset{\text{assoc.}}{\triangleq} a_i\, b_k\, \boldsymbol{e_i} \times \boldsymbol{e_k} \overset{\text{orthon.}}{\triangleq} a_i\, b_k\, \varepsilon_{ikl}\, \boldsymbol{e_l}$$

$$c_l \overset{\text{orthon.}}{\triangleq} \varepsilon_{ikl}\, a_k\, b_l$$

- Triple product (determination of the box volume):

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (a_i\, \boldsymbol{e_i} \times b_k\, \boldsymbol{e_k}) \cdot c_l\, \boldsymbol{e_l} \overset{\text{assoc.}}{\triangleq} a_i\, b_k\, c_l (\boldsymbol{e_i} \times \boldsymbol{e_k}) \cdot \boldsymbol{e_l} \overset{\text{orthon.}}{\triangleq} \varepsilon_{ikl}\, a_i\, b_k\, c_l$$

Revision of Tensor Algebra, Definitions

- Scalar product of tensors (rank two):

$$\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{T} = a_i\, \boldsymbol{e_i} \cdot\; T_{kl}\, \boldsymbol{e_k}\boldsymbol{e_l} \overset{\text{assoc.}}{\triangleq} a_i\, T_{kl}\, (\boldsymbol{e_i} \cdot \boldsymbol{e_k})\boldsymbol{e_l} \overset{\text{orthon.}}{\triangleq} a_i\, T_{kl}\, \delta_{ik}\, \boldsymbol{e_l} = a_i\, T_{il}\, \boldsymbol{e_l}$$

$$b_i = a_k\, T_{ki} = \textstyle\sum_k a_k\, T_{ki}$$

$$\boldsymbol{b} = \boldsymbol{T} \cdot \boldsymbol{a} = T_{ik}\, \boldsymbol{e_i}\boldsymbol{e_k} \cdot a_l\, \boldsymbol{e_l} \overset{\text{assoc.}}{\triangleq} T_{ik}\, a_l\, \boldsymbol{e_i}(\boldsymbol{e_k} \cdot \boldsymbol{e_l}) \overset{\text{orthon.}}{\triangleq} T_{ik}\, a_l\, \delta_{kl}\, \boldsymbol{e_i} = T_{il}\, a_l\, \boldsymbol{e_i}$$

$$b_i = T_{ik}\, a_k$$

In general, $\boldsymbol{T} \cdot \boldsymbol{a} \neq \boldsymbol{a} \cdot \boldsymbol{T}$!

Only in case of symmetric tensors ($\boldsymbol{T}^{\boldsymbol{T}} = \boldsymbol{T},\ T_{ik} = T_{ki}$): $\boldsymbol{T} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{T}$.

Revision of Tensor Algebra

- The most **important property of tensors** making them ideal to describe physical/materials problems is that these **mathematical objects are invariant against transformations of the coordinate system**.

- Hence, the **tensors** (here $\boldsymbol{a}$ and $\boldsymbol{T}$) **remain exactly the same independent of being used in the non-primed or primed coordinate system**:

$$\boldsymbol{a} = a_i\, \boldsymbol{e}_i = a'_i\, \boldsymbol{e}'_i$$
$$\boldsymbol{T} = T_{ik}\, \boldsymbol{e}_i\, \boldsymbol{e}_k = T'_{ik}\, \boldsymbol{e}'_i\, \boldsymbol{e}'_k$$

- Only the components (here a_i and a'_i or T_{ik} and T'_{ik}) or their representations as matrices are different!

- When applying a rotation $\boldsymbol{e}'_i = R_{ik}\, \boldsymbol{e}_k$ and $\boldsymbol{e}_i = S_{ik}\, \boldsymbol{e}'_k$ to the coordinate system, following rotation matrices can be defined from the following cosines:

$$\cos \measuredangle(\boldsymbol{e}'_i, \boldsymbol{e}_k) = \boldsymbol{e}'_i \cdot \boldsymbol{e}_k = R_{il}\, \boldsymbol{e}_l \cdot \boldsymbol{e}_k \overset{\text{orthon.}}{\triangleq} R_{il}\, \delta_{lk} = R_{ik}$$
$$\cos \measuredangle(\boldsymbol{e}_i, \boldsymbol{e}'_k) = \boldsymbol{e}_i \cdot \boldsymbol{e}'_k = S_{il}\, \boldsymbol{e}'_l \cdot \boldsymbol{e}'_k \overset{\text{orthon.}}{\triangleq} S_{il}\, \delta_{lk} = S_{ik}$$
$$R_{ik} = S_{ki}$$

Revision of Tensor Algebra

- Following transformation rules can be obtained:

$$\boldsymbol{a} = a_i\, \boldsymbol{e}_i = a_i\, S_{ik}\, \boldsymbol{e}'_k = a'_k\, \boldsymbol{e}'_k$$
$$\boldsymbol{a} = a'_i\, \boldsymbol{e}'_i = a'_i\, R_{ik}\, \boldsymbol{e}_k = a_k\, \boldsymbol{e}_k$$

$$a'_i = S_{ki}\, a_k = R_{ik}\, a_k \text{ or } a_k = R_{ik}\, a'_i = S_{ik}\, a'_i \quad (\boldsymbol{a}' = \boldsymbol{R} \cdot \boldsymbol{a})$$

$$\boldsymbol{T} = T_{ik}\, \boldsymbol{e}_i\, \boldsymbol{e}_k = T'_{ik}\, \boldsymbol{e}'_i\, \boldsymbol{e}'_k \quad (\boldsymbol{T}' = \boldsymbol{R} \cdot \boldsymbol{T} \cdot \boldsymbol{R}^{\boldsymbol{T}})$$

$$T'_{ik} = R_{il}\, R_{km}\, T_{lm}$$

Revision of Tensor Algebra

- There are quantities which do not change upon transformation of the coordinate system. The so-called invariants are of importance regarding the physical interpretation of tensor problems. The easiest example for an invariant is the length of a vector. Linear combinations of invariants remain invariant!

$$a^2 = a_i' \, a_i' = R_{ik} \, a_k \, R_{il} \, a_l = \delta_{kl} \, a_k \, a_l = a_l \, a_l$$

$$I_1 = T_{ii}$$

$$I_2 = \frac{1}{2} \, T_{ik} \, T_{ki}$$

$$I_3 = \frac{1}{3} \, T_{ik} \, T_{kl} \, T_{li}$$

- **Since the energy of systems cannot change when using different coordinate systems, all potentials must always depend only on tensor invariants!**

Displacement Vector

- First, we introduce the displacement vector as the shortest vector connection from the initial $\boldsymbol{r}$ to the final position $\boldsymbol{r}'$ of a point. Of course, this displacement vector contains both, information about the deformation of the solid and its rigid body motion:

$$\boldsymbol{u} = \boldsymbol{r}' - \boldsymbol{r}$$

$$u_i = x_i' - x_i$$

Strain Tensor

- In order to separate deformation, a tensor of rank two is introduced which describes the relative changes in instantaneous lengths:

$$\mathrm{d}l^2 = \mathrm{d}{x_i}^2$$

$$\mathrm{d}l'^2 = \mathrm{d}x_i'^2 = \left(\mathrm{d}{x_i}^2 + \mathrm{d}{u_i}^2\right)$$

$$\mathrm{d}l'^2 = \mathrm{d}l^2 + 2\,\varepsilon_{ik}\,\mathrm{d}x_i\,\mathrm{d}x_k \text{ with } \varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right)$$

- **Following linear approximation by leaving out the last term is typically used:**

$$\varepsilon_{ik} \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

Strain Tensor

- The (linear) strain tensor is **symmetric**: $\varepsilon_{ik} = \varepsilon_{ki}$ (the sum of the displacement gradients is commutative).
- Hence, it exists a **principle system**, the non-diagonal elements of which are zero. The strain state can be described by the **three principle strains** in the principle system $\varepsilon^{(1)}$, $\varepsilon^{(2)}$ & $\varepsilon^{(3)}$ with:

$$\mathrm{d}x_1' = \left(1 + \varepsilon^{(1)}\right) \mathrm{d}x_1, \text{ etc.}$$

- The **trace** describes the **volume change** by the deformation:

$$\begin{aligned} \mathrm{d}V' &= \mathrm{d}x_1' \, \mathrm{d}x_2' \, \mathrm{d}x_3' \\ &= \mathrm{d}V \left(1 + \varepsilon^{(1)} + \varepsilon^{(2)} + \varepsilon^{(3)}\right) = \mathrm{d}V \left(1 + \varepsilon_{ii}\right) \end{aligned}$$

Strain Tensor

- The principle strains can be obtained by solving the following **eigenvalue problem** (the projection of the strain tensor on the principle system scales with the strength of the principle strains):

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{x} = \varepsilon^{(i)} \, \boldsymbol{x}$$

- Hence, for any non-trivial system $\boldsymbol{x} \neq \boldsymbol{0}$, the **characteristic polynomial** can be solved:

$$\det\left(\boldsymbol{\varepsilon} - \varepsilon^{(i)} \cdot \boldsymbol{1}\right) = 0$$

- The principle strains are of course invariants of the tensor because they don't change under transformation of the coordinate system.

Rotation Tensor

- In analogy to the derivation of the strain tensor, the rotations can also be described in the form of a tensor of rank two:

$$\omega_{ik} \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right)$$

antisymmetric $\omega_{ik} = -\omega_{ki}$

Stress Tensor

- The force acting on any volume is:

$$\int F_i \, \mathrm{d}V$$

Stress Tensor

- Hence, there is a tensor of rank two, σ_{ik}, the gradient of which corresponds to the forces $F_i = \frac{\partial \sigma_{ik}}{\partial x_k}$.
- The volume integral then can be converted by using Gauss' theorem into a surface integral over the tensor itself:

$$\int F_i \, \mathrm{d}V = \int \frac{\partial \sigma_{ik}}{\partial x_k} \, \mathrm{d}V \overset{\text{Gauss}}{\triangleq} \oint \sigma_{ik} \, \mathrm{d}n_k$$

- It's of fundamental importance, that this can be done. The continuum theory requests that the microscopic interaction is of short range (atomic scale). Hence, different/any volumes must exclusively interact via their surfaces. This is obviously possible if the forces can be (fully) expressed in terms of the above introduced tensor.

Stress Tensor

- The torque acting on the volume is:

$$\int F_i\, x_k - F_k\, x_i \,\mathrm{d}V$$

- A similar conversion leads to:

$$\int \frac{\partial \sigma_{il}}{\partial x_l} x_k - \frac{\partial \sigma_{kl}}{\partial x_l} x_i \,\mathrm{d}V$$

$$\overset{\text{part. int.}}{\triangleq} \int \frac{\partial(\sigma_{il}\, x_k - \sigma_{kl} x_i)}{\partial x_l} \mathrm{d}V - \int \left(\sigma_{il} \frac{\partial x_k}{\partial x_l} - \sigma_{kl} \frac{\partial x_i}{\partial x_l} \right) \mathrm{d}V$$

$$\overset{\text{Gauss}}{\triangleq} \oint \sigma_{il}\, x_k - \sigma_{kl} x_i \,\mathrm{d}n_l + \int (\sigma_{ki} - \sigma_{ik}) \,\mathrm{d}V$$

Stress Tensor

- Since torques also have to be transmitted via the surface only, the volume integral term must vanish:

$$\int (\sigma_{ki} - \sigma_{ik})\,\mathrm{d}V = 0 \text{ by } \sigma_{ik} = \sigma_{ki}$$

- The stress tensor is symmetric. Important: this is often misinterpreted in the way that torque balance is automatically fulfilled. It only means that the torques in any volume element are transmitted via the surfaces of the volumes, no more, no less.

Stress Tensor

- Hence, there is a **principle system** with the principle stresses $\sigma^{(1)}$, $\sigma^{(2)}$ and $\sigma^{(3)}$.
- The **characteristic polynomial** is:

$$\det\left(\boldsymbol{\sigma} - \sigma^{(i)} \cdot \mathbf{1}\right) = 0$$

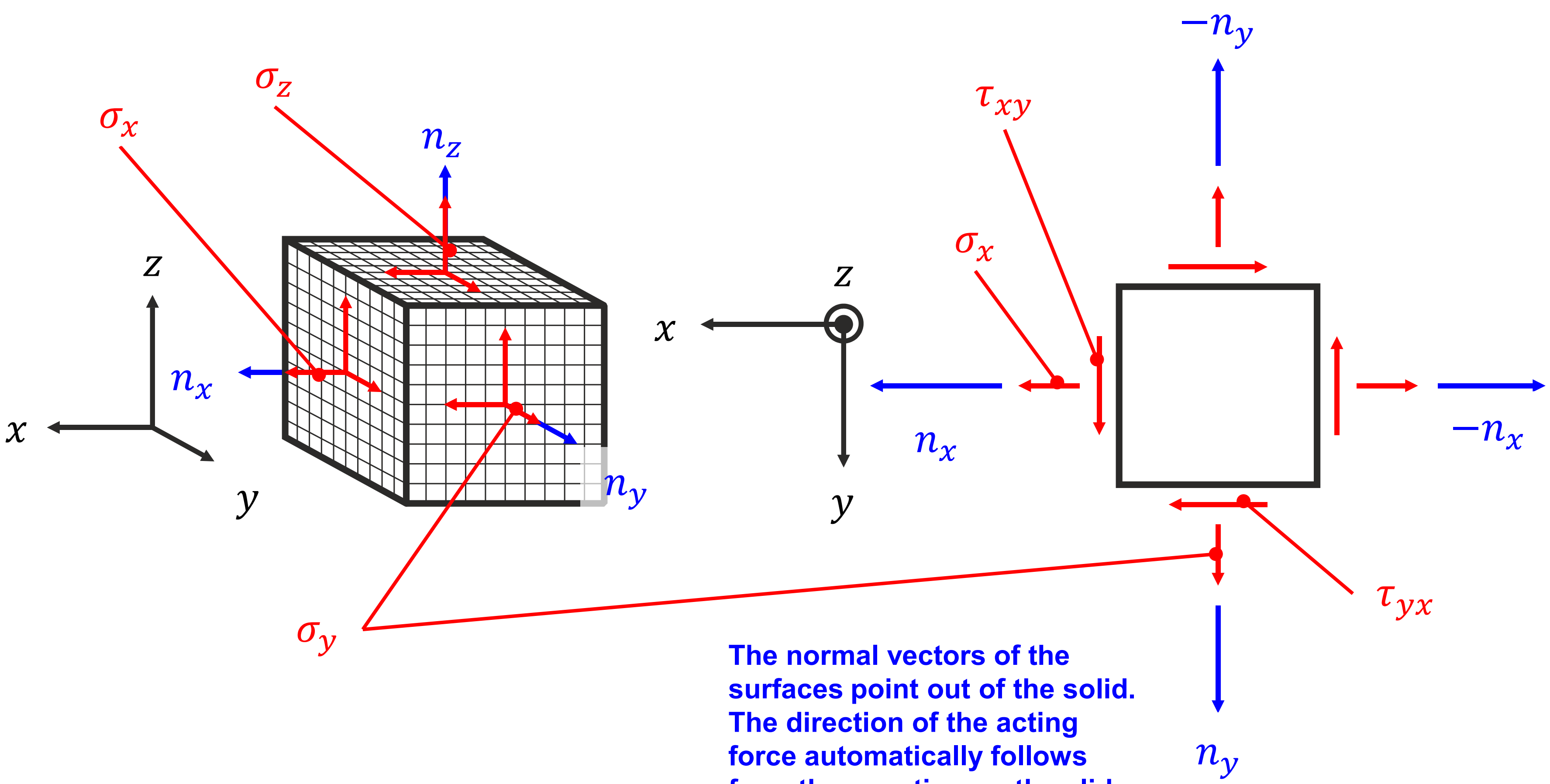
- The principle stresses are invariants of the stress tensor.

Stress Vector

- The stress vector $\boldsymbol{t}$ can be calculated by the projection of the stress tensor onto a normal vector of the respective plane $\boldsymbol{n}$:

$$t_i = n_k \, \sigma_{ik}$$

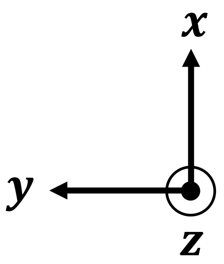
Reference Frame



The normal vectors of the surfaces point out of the solid. The direction of the acting force automatically follows from the equation on the slide before.

Examples: Rotation

- Rotation of the system:



$$(x_i) = \begin{pmatrix}1\\0\\0\end{pmatrix},\ (y_i) = \begin{pmatrix}0\\1\\0\end{pmatrix},\ (z_i) = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

rotation of 45° about z: $(R_{ik}) = \begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ 0 & 0 & 1\end{pmatrix}$

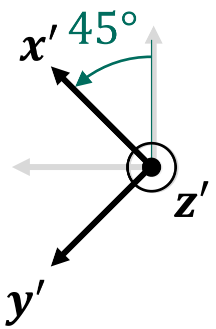
obtained by

$\cos\sphericalangle(\boldsymbol{x}',\boldsymbol{x}) = \cos\sphericalangle(\boldsymbol{y}',\boldsymbol{y})$

$= \cos\sphericalangle(\boldsymbol{x}',\boldsymbol{y}) = \cos 45° = \frac{\sqrt{2}}{2}$

$\cos\sphericalangle(\boldsymbol{z}',\boldsymbol{z}) = \cos 0° = 1$

$\cos\sphericalangle(\boldsymbol{y}',\boldsymbol{x}) = \cos 135° = -\frac{\sqrt{2}}{2}$

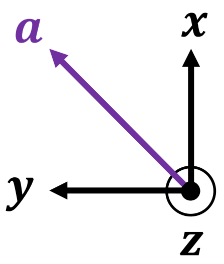


$$(x_i') = (R_{ik}\,x_k) = \begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\\ 0\end{pmatrix},\ (y_i') = \begin{pmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\\ 0\end{pmatrix},\ (z_i') = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Description of the base vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$ and $\boldsymbol{z}'$ in the reference frame $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$

Examples: Rotation

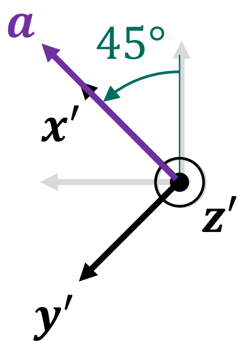
- Rotation of the system:



$$(a_i) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\boldsymbol{a}$ remains the same irrespective of the coordinate system

rotation of $45°$ about z: $(R_{ik}) = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

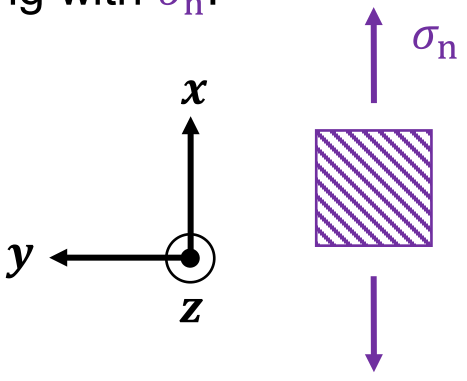


$$(a_i') = (R_{ik}\, a_k) = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

Description of the vector $\boldsymbol{a}$ in the frame $\boldsymbol{x}'$, $\boldsymbol{y}'$ and $\boldsymbol{z}'$.

Examples: Uniaxial Loading

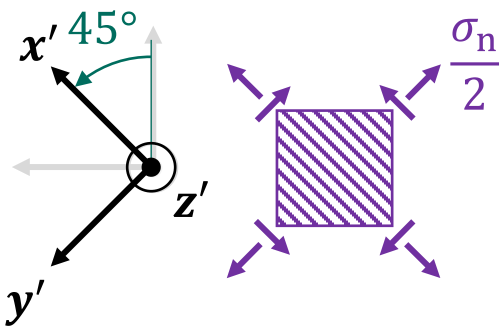
- Uniaxial loading with σ_n:



$$(\sigma_{ik}) = \begin{pmatrix} \sigma_\text{n} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

σ remains the same irrespective of the coordinate system

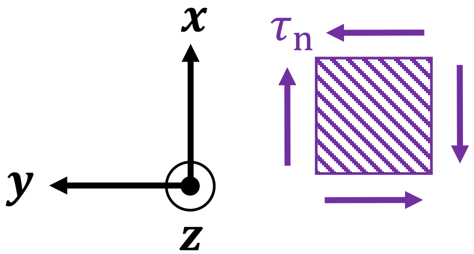
rotation of 45° about z: $$(R_{ik}) = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$(\sigma'_{ik}) = \begin{pmatrix} \frac{\sigma_\text{n}}{2} & -\frac{\sigma_\text{n}}{2} & 0 \\ -\frac{\sigma_\text{n}}{2} & \frac{\sigma_\text{n}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Examples: Shear Loading

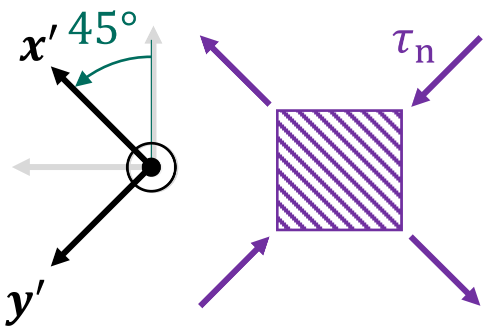
- Uniaxial loading with σ_n:



$$(\sigma_{ik}) = \begin{pmatrix} 0 & \tau_\mathrm{n} & 0 \\ \tau_\mathrm{n} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

σ remains the same irrespective of the coordinate system

rotation of 45° about z: $(R_{ik}) = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

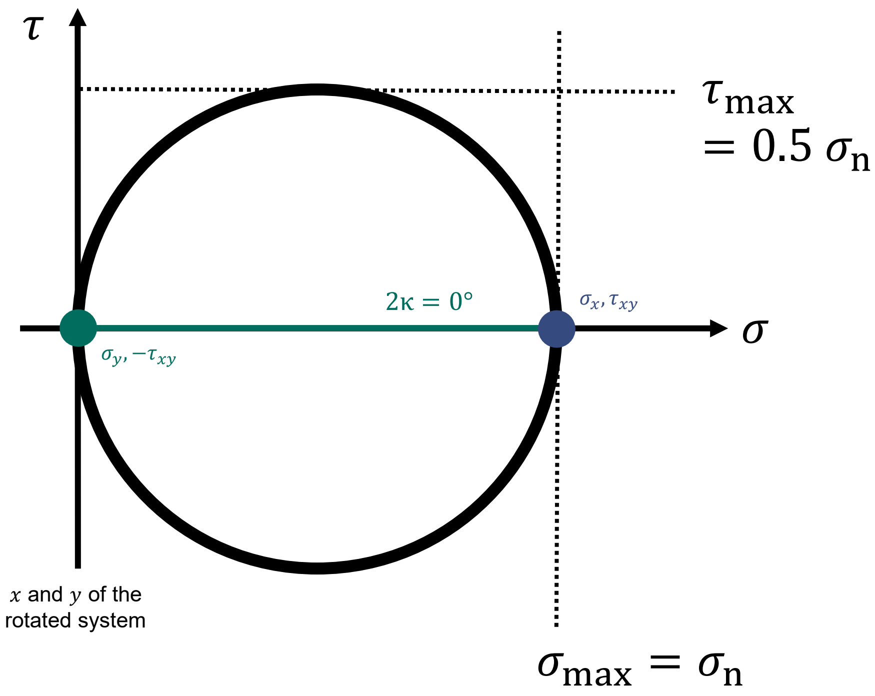


$$(\sigma_{ik})' = \begin{pmatrix} \tau_\mathrm{n} & 0 & 0 \\ 0 & -\tau_\mathrm{n} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

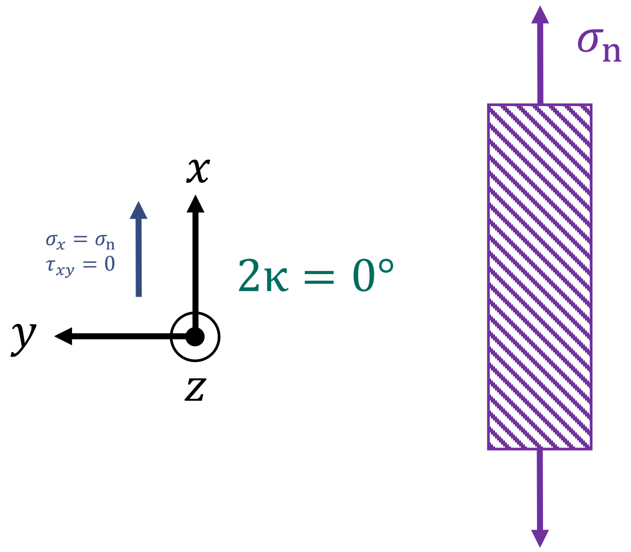
Examples: Uniaxial Stress

- Uniaxial loading with σ_n:

visualization with Mohr's circle



rotation of κ about z



$$(\sigma_{ik}) = \begin{pmatrix} \sigma_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Examples: Uniaxial Stress

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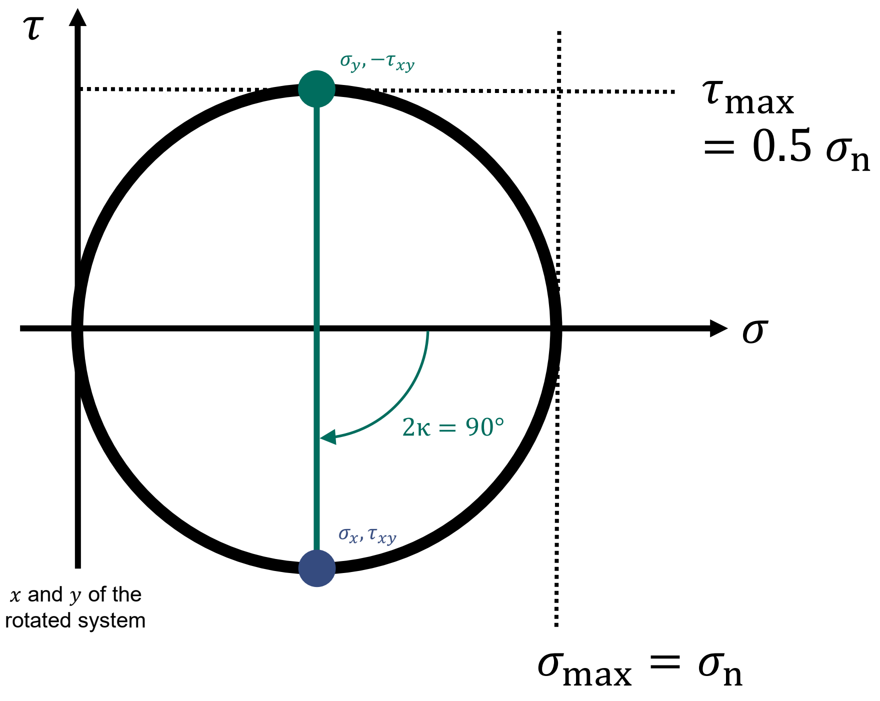
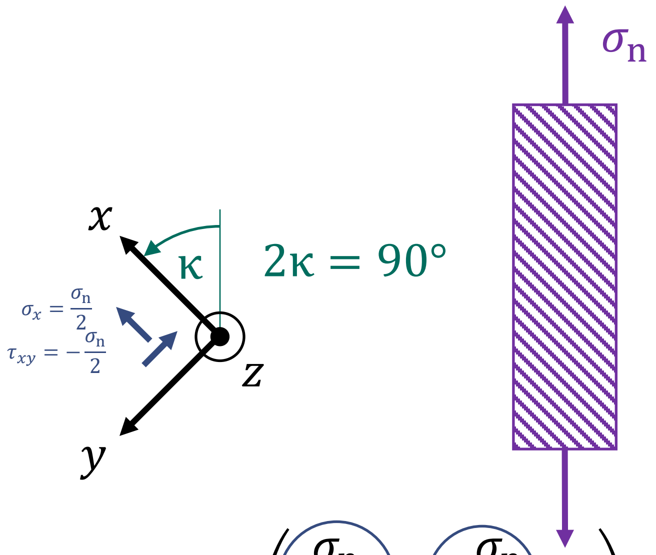
visualization with Mohr's circle



rotation of κ about z



$$(\sigma_{ik}) = \begin{pmatrix} \frac{\sigma_\mathrm{n}}{2} & -\frac{\sigma_\mathrm{n}}{2} & 0 \\ -\frac{\sigma_\mathrm{n}}{2} & \frac{\sigma_\mathrm{n}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Examples: Uniaxial Stress

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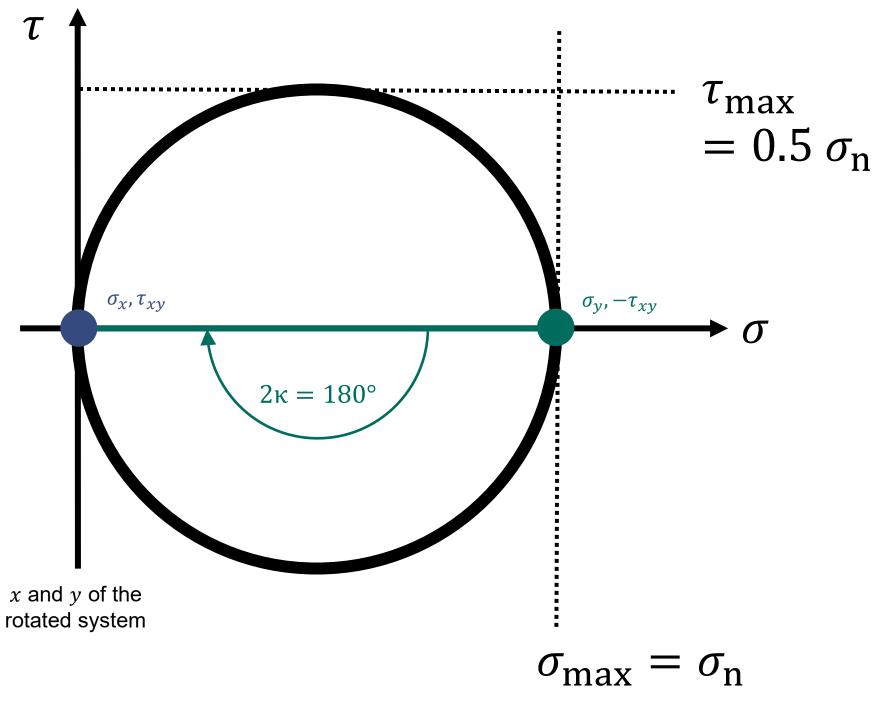
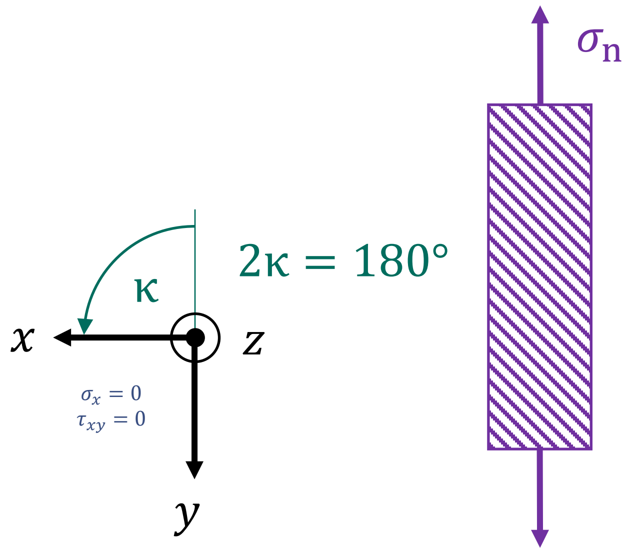


rotation of κ about z

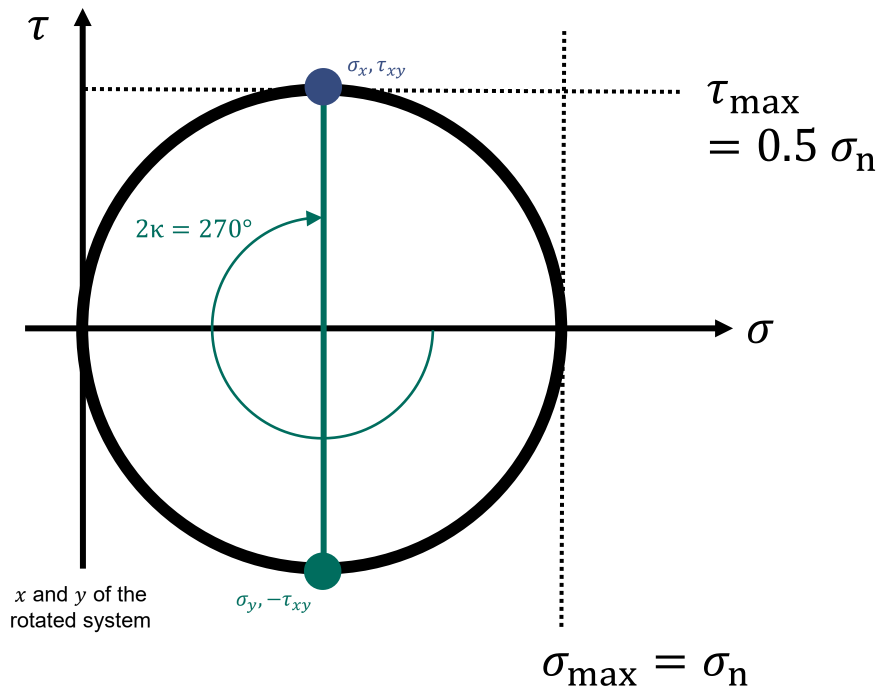


$$(\sigma_{ik}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_n & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

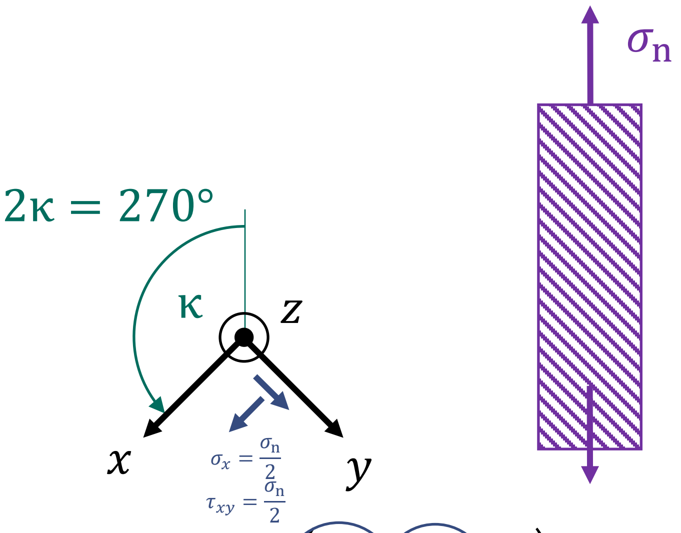
Examples: Uniaxial Stress

- Uniaxial loading with σ_n:

visualization with Mohr's circle



rotation of 2κ about z

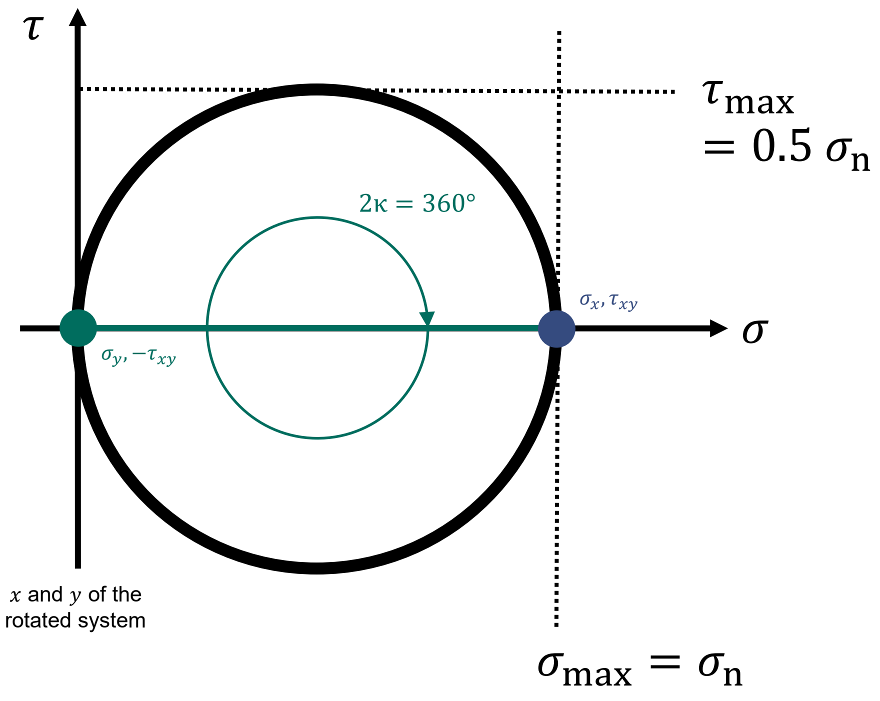


$$(\sigma_{ik}) = \begin{pmatrix} \frac{\sigma_n}{2} & \frac{\sigma_n}{2} & 0 \\ \frac{\sigma_n}{2} & \frac{\sigma_n}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

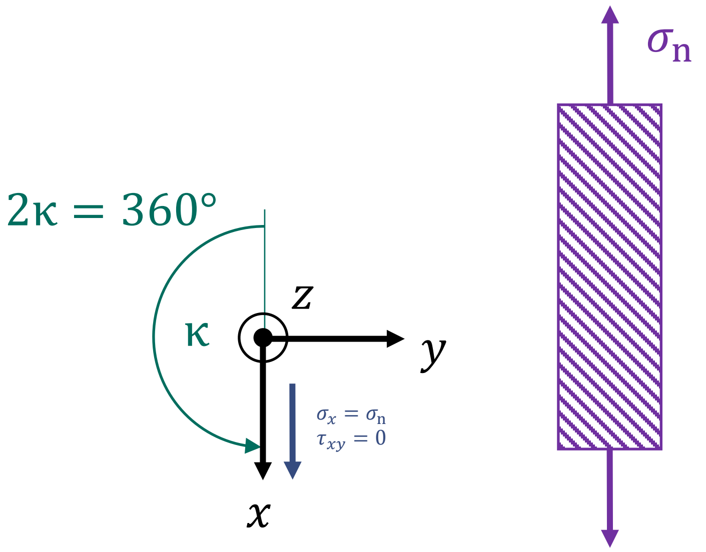
Examples: Uniaxial Stress

- Uniaxial loading with σ_n:

visualization with Mohr's circle



rotation of κ about z

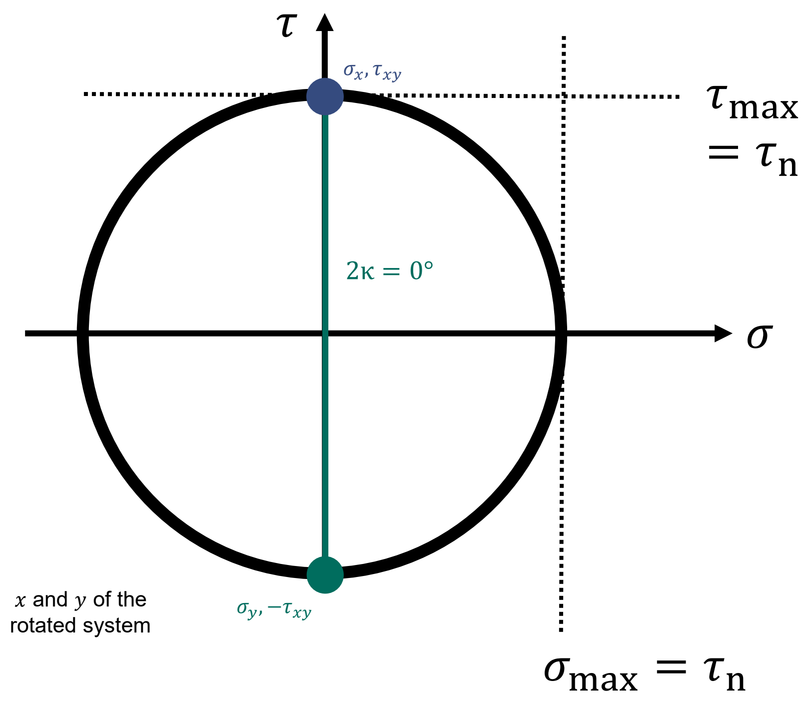


$$(\sigma_{ik}) = \begin{pmatrix} \sigma_\mathrm{n} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

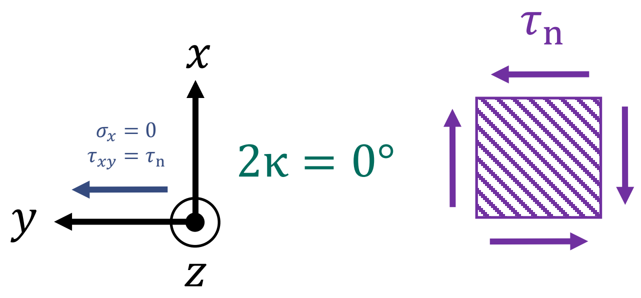
Examples: Shear Loading

- Pure shear with τ_n:

visualization with Mohr's circle



rotation of κ about z



$$(\sigma_{ik}) = \begin{pmatrix} 0 & \tau_\mathrm{n} & 0 \\ \tau_\mathrm{n} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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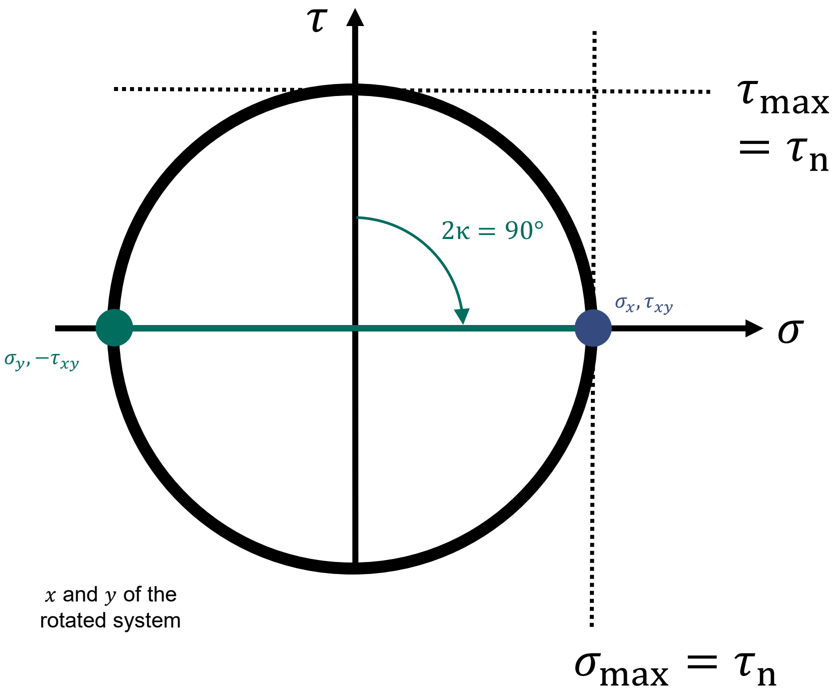
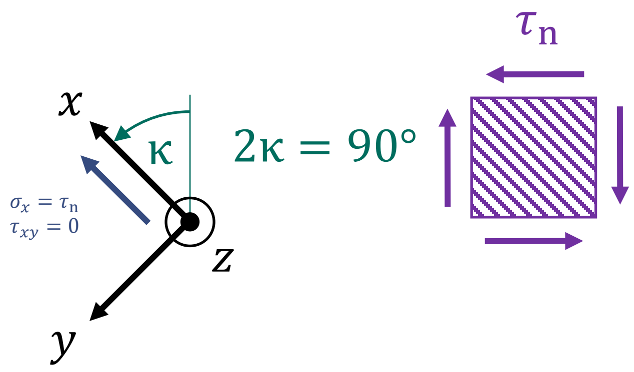


rotation of κ about z



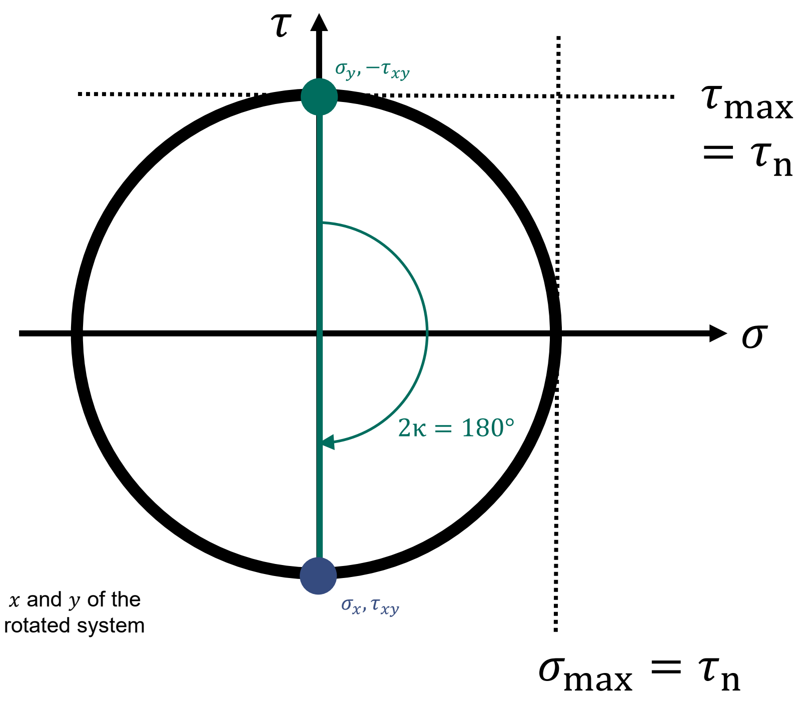
$$(\sigma_{ik}) = \begin{pmatrix} \tau_\mathrm{n} & 0 & 0 \\ 0 & -\tau_\mathrm{n} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

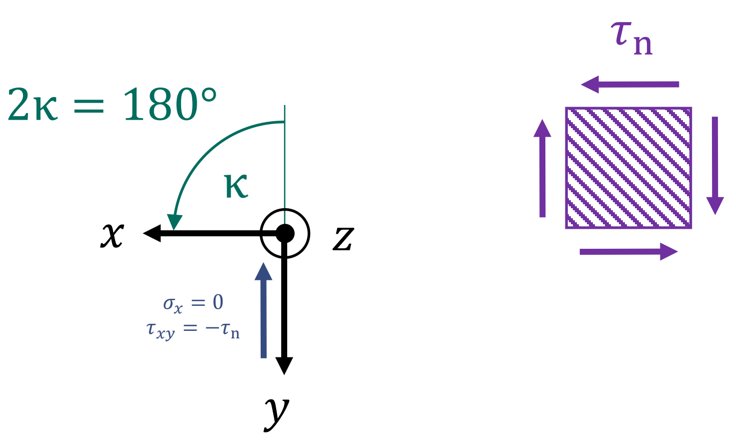
Examples: Shear Loading

- Pure shear with τ_n:

rotation of κ about z

visualization with Mohr's circle



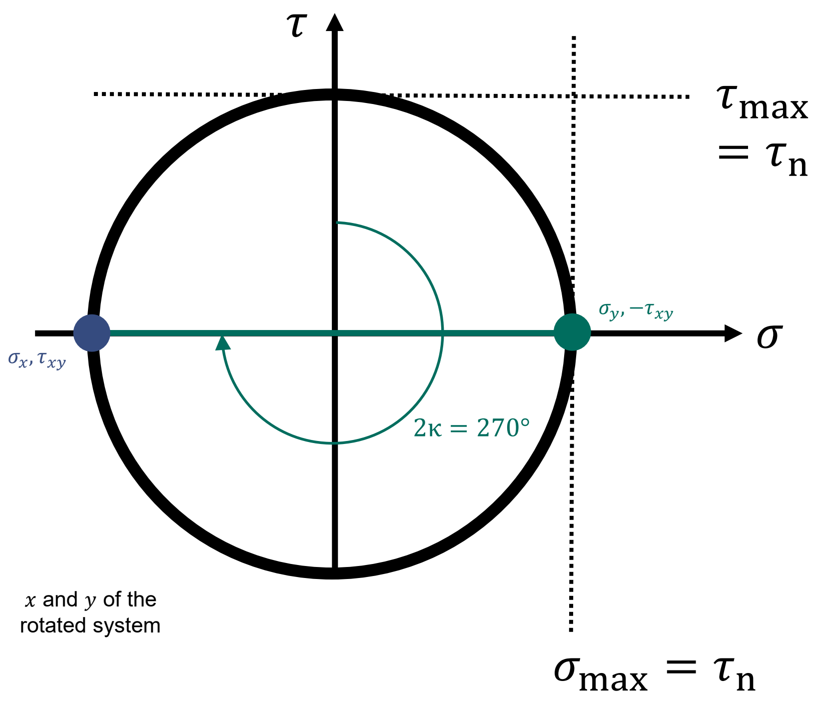


$$(\sigma_{ik}) = \begin{pmatrix} 0 & -\tau_n & 0 \\ -\tau_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

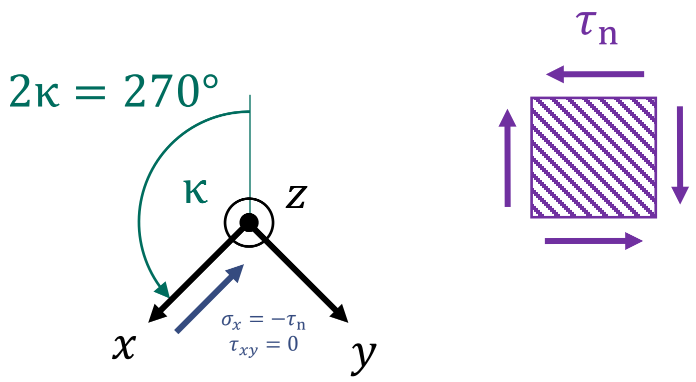
Examples: Shear Loading

- Pure shear with τ_n:

visualization with Mohr's circle



rotation of κ about z

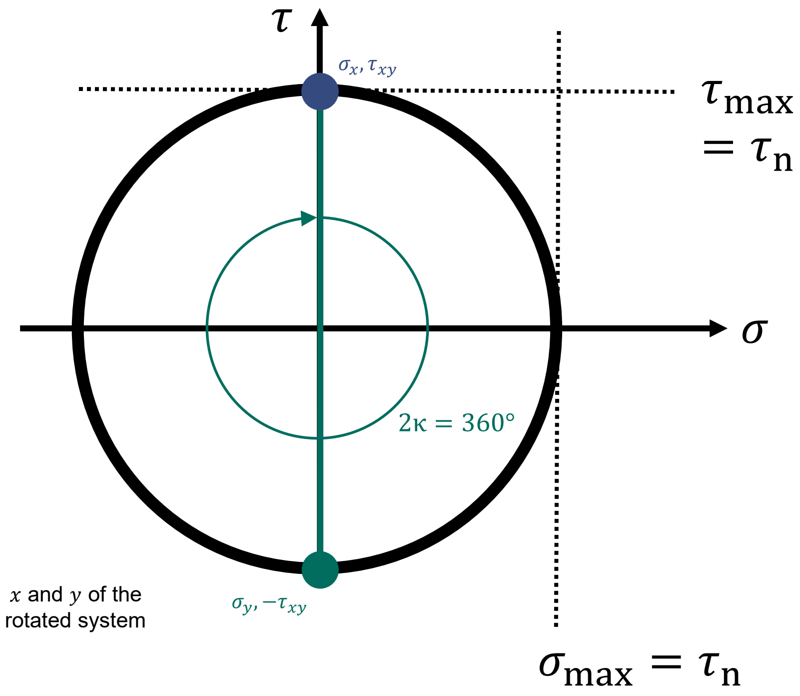


$$(\sigma_{ik}) = \begin{pmatrix} -\tau_\mathrm{n} & 0 & 0 \\ 0 & \tau_\mathrm{n} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

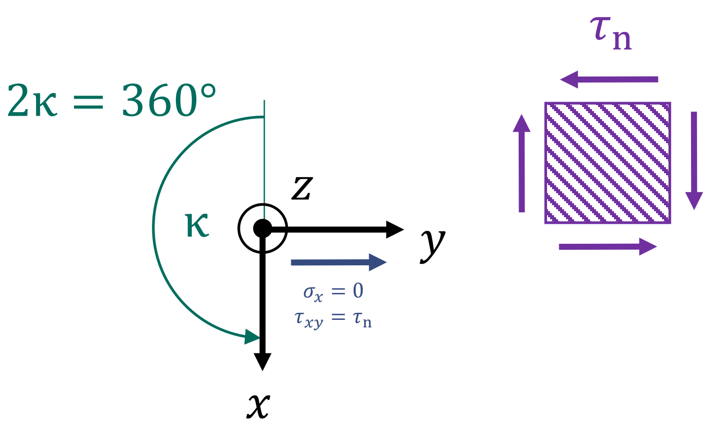
Examples: Shear Loading

- Pure shear with τ_n:

visualization with Mohr's circle



rotation of κ about z



$$(\sigma_{ik}) = \begin{pmatrix} 0 & \tau_n & 0 \\ \tau_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Equilibrium

- In equilibrium, the internal forces compensate in each volume:

$$F_i = 0$$

- The equilibrium is therefore determined by solving:

$$\frac{\partial \sigma_{ik}}{\partial x_k} = 0$$

Hooke's Law

- Solving $\frac{\partial \sigma_{ik}}{\partial x_k} = 0$ is not useful since boundary conditions might not be available in a proper way. Therefore, a conversion to strains and, finally, to displacements is useful.
- The conversion needs the introduction of suitable materials laws. The simplest form is as follows:

$$\sigma_{ik} = C_{iklm}\, \varepsilon_{lm}$$
$$\varepsilon_{ik} = S_{iklm}\, \sigma_{lm}$$

- Hence, $\frac{\partial}{\partial x_k}(C_{iklm}\, \varepsilon_{lm}) = 0$ and $\frac{\partial}{\partial x_k}\left(C_{iklm}\left(\frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial x_l}\right)\right) = 0$ follow.

Compliance and Stiffness

- The materials property has $3 \cdot 3 \cdot 3 \cdot 3 = 81$ coefficients $6 \cdot 6 = 36$ of which are independent coefficients considering the symmetries of the stress and strain tensors:

$$C_{iklm} = C_{kilm}$$
$$C_{iklm} = C_{ikml}$$

- The principle symmetry is more difficult to prove by application of Schwarz's theorem. Anyway, the number of independent coefficients further reduces to 21:

$$C_{iklm} = C_{lmik}$$

Additional Information: Voigt Notation

- The symmetry properties allow for a more efficient way of writing down the numbers by converting pairs of indices as follows:

$$
\begin{aligned}
11 &\rightarrow 1 \\
22 &\rightarrow 2 \\
33 &\rightarrow 3 \\
23 &\rightarrow 4 \\
13 &\rightarrow 5 \\
12 &\rightarrow 6
\end{aligned}
$$

- Very important: The Voigt-converted matrices and vectors aren't tensors! The transformation laws presented in the first slides cannot be applied to these things.

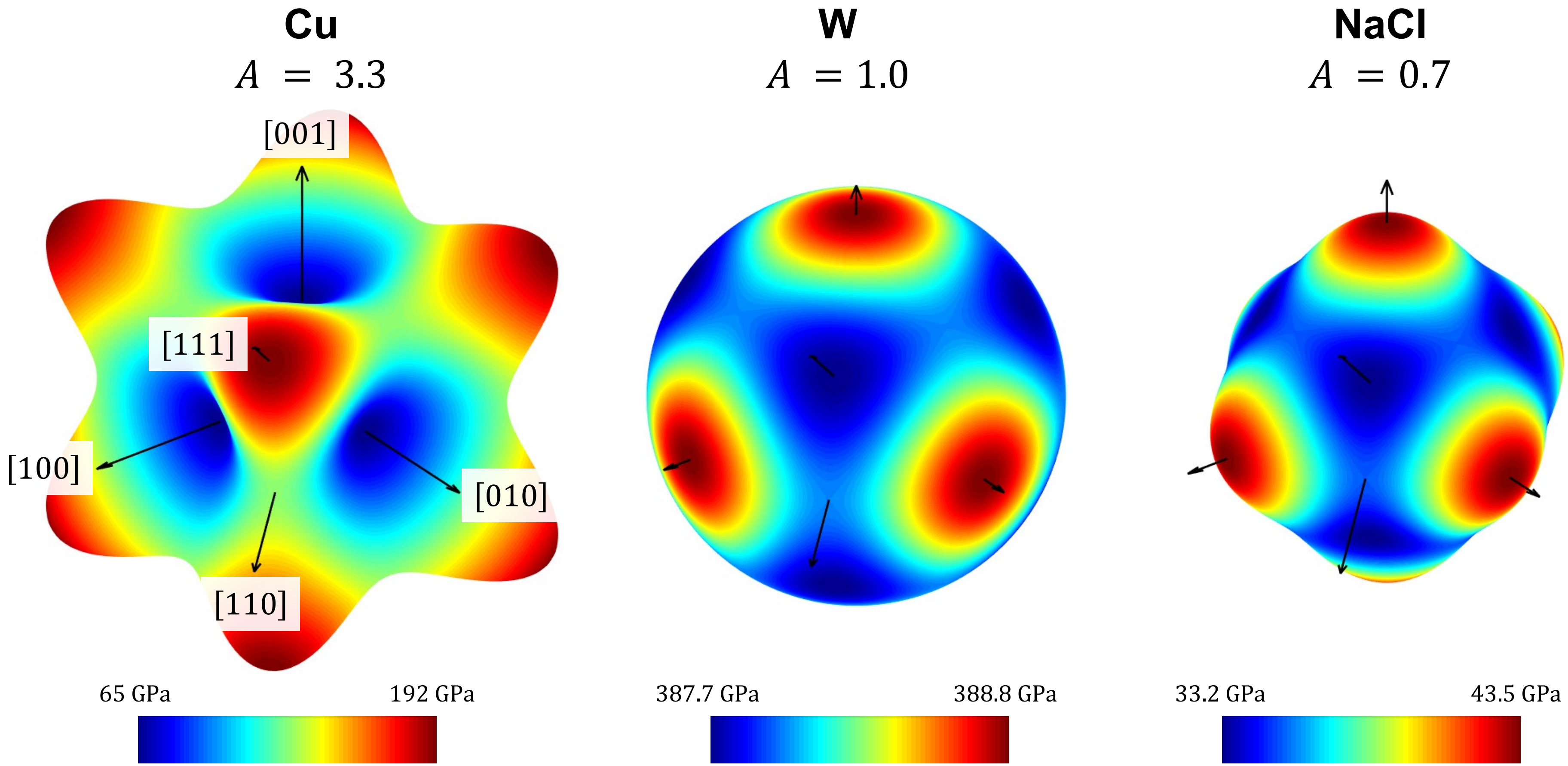
Compliance and Stiffness

material	prototype	Struktur-bericht	C_{11} / GPa	C_{12} / GPa	C_{44} / GPa	A
Cu	Cu	A1	**166**	**120**	**76**	**3.3**
Al			106	60	28	1.2
Au			193	164	42	2.9
Ni			251	150	124	2.5
α-Fe	W	A2	230	135	117	2.5
W			**501**	**198**	**151**	**1.0**
Si	diamond	A4	166	64	80	1.6
NaCl	NaCl	B1	**49**	**13**	**13**	**0.7**

C. Teodosiu: “Elastic Models of Crystal Defects”, Berlin, Heidelberg: Springer (1982)
R. E. Newnham: “Properties of materials“, Oxford, UK: Oxford University Press (2005)
G. E. Dieter: “Mechanical Metallurgy“, London, etc.: McGraw-Hill (1988)

Compliance and Stiffness

- Visualization of the anisotropic Young's moduli of single crystals deformed uniaxial along certain axes:



Compliance and Stiffness

material	prototype	Struktur-bericht	C_{11} / GPa	C_{12} / GPa	C_{13} / GPa	C_{33} / GPa	C_{44} / GPa
Mg	Mg	A3	59	26	21	62	16
Zn			164	36	53	63	39
α-Ti			41	35	29	53	7

C. Teodosiu: “Elastic Models of Crystal Defects”, Berlin, Heidelberg: Springer (1982)

Isotropic, Linear-Elastic Material

- The potential energy of an elastically loaded material can only depend on invariants of the strain tensor:

$$U = C_1 {\varepsilon_{ii}}^2 + C_2 \left(\frac{1}{2}\ \varepsilon_{ik}\ \varepsilon_{ki}\right) \text{ with } \sigma_{ik} = \frac{\partial U}{\partial \varepsilon_{ik}}$$

(A form which results in a linear elastic materials law.)

- Hence, the corresponding isotropic material has only two independent elastic coefficients:

$$\sigma_{ik} = 2C_1 \varepsilon_{ll}\ \delta_{ik} + C_2\ \varepsilon_{ik}$$

Isotropic, Linear-Elastic Material

- The equation can be expressed in terms of simple, elastic properties, for example G and ν:

$$\sigma_{ik} = 2G\,\varepsilon_{ik} + \frac{2G\,\nu}{1-2\nu}\,\delta_{ik}\varepsilon_{ll}$$

$$C_{iklm} = \frac{2G(1-\nu)}{3(1-2\nu)}\delta_{ik}\,\delta_{lm} + G\left(\delta_{il}\,\delta_{km} + \delta_{im}\,\delta_{kl} - \frac{2}{3}\,\delta_{ik}\,\delta_{lm}\right)$$

- There are also other common pairs of properties, like G and K in solid state physics or E and ν in mechanical engineering. G and ν are often used for the description of dislocations.

Isotropic, Linear-Elastic Material

- Isotropic behavior in polycrystals is obtained for a random distribution of crystal orientations.
- There are many different ways of averaging over the orientation distribution function $f(\boldsymbol{g})$ (random for $f(\boldsymbol{g}) = 1$).
- Simple cases are parallel or serial with homogeneous strain (Voigt) or stress (Reuss), respectively, within the grains.

Isotropic, Linear-Elastic Material

material	prototype	Struktur-bericht	ρ / kg/m^3	c_T / m/s	c_L / m/s	G / GPa	M / GPa	ν / 1	E / GPa
Cu	Cu	A1	8933	2325	4759	48	202	0.34	130
Al			2698	3111	6374	26	110	0.34	70
Au			19281	1200	3240	28	202	0.42	79
Ni			8907	2929	5608	76	280	0.31	201
α-Fe	W	A2	7873	3224	5957	82	279	0.29	212
W			19254	2887	5221	160	525	0.28	411
Mg	Mg	A3	1738	3163	5823	17	59	0.29	45
Zn			7135	2421	4187	42	125	0.25	104
α-Ti			4508	3128	6130	44	169	0.32	117
Si	diamond	A4	2329	–	8433	80	166	0.22	145
NaCl	NaCl	B1	2589	2772	5584	20	81	0.34	53

$M = \frac{2G(1-\nu)}{1-2\nu}$ denotes an elastic modulus associated to the propagation of pressure waves (longitudinal).

G. W. C. Kaye & T. H. Laby: "Tables of Physical and Chemical Constants", Essex, England; New York: Longman (1995)
M. Matsui: "Simultaneous sound velocity and density measurements of NaCl at high temperatures and pressures: Application as a primary pressure standard", American Mineralogist 97 (2012) 1670-1675

Summary

- The simplest description of the deformation of solids is **linear elasticity with interaction forces of short range**. Differential equations of the form $\frac{\partial \sigma_{ik}}{\partial x_k} = 0$ have to be solved. Assuming an anisotropic materials law and using the linearized strain tensor, differential equations of the displacement vector can be used.
- **Crystalline materials are anisotropic**. In case of **elastic isotropy** in polycrystals, **two independent quantities** are necessary to describe the linear elastic behavior.