

A simple derivation of the Kramers-Kronig relations from the perspective of system theory

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For a linear and time-invariant system the property of causality translates from the requirement of a vanishing impulse response for times smaller zero in time domain directly to the Kramers-Kronig relations in frequency domain.

Proof:

Let $h(t)$ be the impulse response of a linear and time-invariant system. The frequency response of this system is then given by its Fourier-transform

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt, \quad (1.1)$$

which for an electrical system corresponds to its impedance. Assuming causality¹ of the described system,

$$h(t) = 0 \quad \forall t < 0 \quad (1.2)$$

holds and the calculation of the frequency response simplifies to

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} h(t) e^{-j\omega t} dt. \quad (1.3)$$

This again is equivalent to

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) \sigma(t) e^{-j\omega t} dt \quad (1.4)$$

with $\sigma(t)$ denoting the Heaviside step function. As the right side of the equation denotes the Fourier-transform

$$F\{h(t) \cdot \sigma(t)\}, \quad (1.5)$$

the convolution theorem can be applied resulting in

$$F\{h(t) \cdot \sigma(t)\} = \frac{H(\omega) * F\{\sigma(t)\}}{\sqrt{2\pi}}. \quad (1.6)$$

¹ No system response before an excitation of the system has taken place.

By evaluating the Fourier transformation of the Heaviside step function one finally finds

$$H(\omega) = H(\omega) * \left(\frac{1}{2\pi} \left(\frac{1}{j\omega} + \pi\delta(\omega) \right) \right) \quad (1.7)$$

as an equality being fulfilled for linear and time-invariant systems, which are causal in addition.

Equation (1.7) claims an invariance of the frequency response under convolution with the Fourier transform of the Heaviside step function, which will turn out to be equivalent to the fulfillment of the Kramers-Kronig relations. In other words: The specific relation between real and imaginary part of the frequency response described by Kramers-Kronig guarantees that Equation (1.7) holds. Thus, Equation (1.7) can be considered an early or raw form of the Kramers-Kronig relations.

In the following, Equation (1.7) will be further simplified, until the Kramers-Kronig relations are finally being obtained. Executing the convolution leads to

$$H(\omega) = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \frac{H(\omega')}{j(\omega - \omega')} d\omega' + \int_{-\infty}^{\infty} \pi \cdot H(\omega') \cdot \delta(\omega - \omega') d\omega' \right) \quad (1.8)$$

respectively

$$H(\omega) = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \frac{H(\omega')}{j(\omega - \omega')} d\omega' + \pi \cdot H(\omega) \right). \quad (1.9)$$

By isolating $H(\omega)$, Equation (1.10) can be obtained:

$$H(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{H(\omega')}{j(\omega - \omega')} d\omega' \quad (1.10)$$

It describes the frequency response of causal systems being invariant under a Hilbert-transformation. From that the Kramers Kronig relations can be easily obtained by rewriting Equation (1.10) for only positive bounds of integration:

The integral is split up

$$\int_{-\infty}^{\infty} \frac{H(\omega')}{j(\omega - \omega')} d\omega' = \int_0^{\infty} \frac{H(\omega')}{j(\omega - \omega')} d\omega' + \int_{-\infty}^0 \frac{H(\omega')}{j(\omega - \omega')} d\omega' \quad (1.11)$$

and the negative bounded integral is slightly rewritten to

$$\int_{-\infty}^0 \frac{H(\omega')}{j(\omega - \omega')} d\omega' = \int_0^{\infty} \frac{H(-\omega'')}{j(\omega + \omega'')} d\omega'' \quad (1.12)$$

From the definition of the Fourier Integral it follows directly, that

$$H(-\omega'') = H^*(\omega'') \quad (1.13)$$

holds and therefore (1.11) can be written as

$$H(\omega) = \frac{1}{\pi} \left(\int_0^{\infty} \frac{H(\omega')}{j(\omega - \omega')} d\omega' + \int_0^{\infty} \frac{H^*(\omega'')}{j(\omega + \omega'')} d\omega'' \right). \quad (1.14)$$

Merging both integrals yields

$$H(\omega) = \frac{1}{\pi} \int_0^{\infty} \frac{H(\omega') \cdot (\omega + \omega') + H^*(\omega') \cdot (\omega - \omega')}{j(\omega^2 - \omega'^2)} d\omega'. \quad (1.15)$$

By splitting the frequency response into its real and imaginary part finally the Kramers-Kronig relations are obtained:

$$H(\omega) = \frac{2}{\pi} \int_0^{\infty} \frac{H_{\text{Im}}(\omega') \cdot \omega' - j \cdot H_{\text{Re}}(\omega') \cdot \omega}{(\omega^2 - \omega'^2)} d\omega' \quad (1.16)$$

Commonly, Equation (1.16) is split up into two equations for real and imaginary part

$$H_{\text{Re}}(\omega) + jH_{\text{Im}}(\omega) = \frac{2}{\pi} \int_0^{\infty} \frac{\omega' H_{\text{Im}}(\omega')}{(\omega^2 - \omega'^2)} d\omega' + \frac{2}{\pi} \int_0^{\infty} j \frac{-\omega H_{\text{Re}}(\omega')}{(\omega^2 - \omega'^2)} d\omega', \quad (1.17)$$

which finally leads to the Kramers-Kronig relations in their well-known form²:

$$\begin{aligned} H_{\text{Re}}(\omega) &= \frac{2}{\pi} \int_0^{\infty} \frac{\omega' H_{\text{Im}}(\omega')}{(\omega^2 - \omega'^2)} d\omega' \\ H_{\text{Im}}(\omega) &= \frac{-2}{\pi} \int_0^{\infty} \frac{\omega H_{\text{Re}}(\omega')}{(\omega^2 - \omega'^2)} d\omega' \end{aligned} \quad (1.18)$$

For citation, a brief version of this derivation can be found in [1].

Literature:

- [1] M. Schönleber, D. Klotz and E. Ivers-Tiffée, *A Method for Improving the Robustness of linear Kramers-Kronig Validity Tests*, *Electrochimica Acta* 131, pp. 20-27 (2014), [10.1016/j.electacta.2014.01.034](https://doi.org/10.1016/j.electacta.2014.01.034).

² In the field of dielectric responses the frequency response is defined differently as $\varepsilon(\omega) = \varepsilon_{\text{Re}}(\omega) - j\varepsilon_{\text{Im}}(\omega)$ instead of $H(\omega) = H_{\text{Re}}(\omega) + jH_{\text{Im}}(\omega)$. Using that definition, the sign of the Kramers-Kronig relations changes to

$$\varepsilon_{\text{Re}}(\omega) = \frac{-2}{\pi} \int_0^{\infty} \frac{\omega' \varepsilon_{\text{Im}}(\omega')}{(\omega^2 - \omega'^2)} d\omega' \quad , \quad \varepsilon_{\text{Im}}(\omega) = \frac{2}{\pi} \int_0^{\infty} \frac{\omega \varepsilon_{\text{Re}}(\omega')}{(\omega^2 - \omega'^2)} d\omega'$$